The $:\phi_4^4:$ quantum field theory, I. Wave operator, holomorphity and Wick kernel. *

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Abstract

With the help of the complex structure and the wave operator of the nonlinear classical Klein-Gordon equation with the interaction u_4^4 we define the Wick kernel of the interacting quantum field in four-dimensional space-time and consider its properties. In particular, the diagonal of this Wick kernel is (real) solutions of the classical nonlinear Klein-Gordon equation with the interaction u_4^4.

1. Introduction.

In the paper [22], see also [17, 36], we have constructed the solution[1] of the quantum wave equation

$$\Box\phi(t,x) + m^2\phi(t,x) + \lambda :\phi^3(t,x):= 0 \tag{1.1}$$

in four–dimensional space–time. In the integral form this equation is the following

$$\phi(t,x) = \phi_{in}(t,x) - \lambda \int_{-\infty}^{t}\int R(t-\tau, x-y) :\phi^3(\tau,y): d\tau d^3y, \tag{1.2}$$

$$\phi_{out}(t,x) = \phi_{in}(t,x) - \lambda \int_{-\infty}^{+\infty}\int R(t-\tau, x-y) :\phi^3(\tau,y): d\tau d^3y, \tag{1.3}$$

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[1] The constructed solution is nontrivial [22, Theorem 4.3], [29], but here we do not discuss the question of the nontriviality of the $:\phi_4^4:$ quantum theory [13, 1, 2, 11, 17, 35, 22].

where the in-field ϕ_{in} is the in-coming free quantum field and : : is the Wick normal ordering with respect to the free field ϕ_{in}.

In the papers [17, 22] this solution has been correctly defined as a bilinear form on $D_{coh}(\vartheta) \times D_{coh}(\vartheta)$, where $D_{coh}(\vartheta)$ is a subspace in the Fock space of the in–field ϕ_{in}. The subspace $D_{coh}(\vartheta)$ is generated by linear combinations of coherent vectors which are near to the vacuum.

In the present paper we consider the solution of the quantum nonlinear wave equation (1.1), which was constructed in [17, 22], from the other point of view.

We introduce explicitly the Wick kernel for the interacting field

$$\phi(e_{z_1}, e_{z_2}) = \frac{1}{2}\exp(\langle z_1, z_2\rangle_{H^{1/2}(I\!R^3,\mathbb{C})})(RW_{in}R^{-1}(\bar{z}_1 + z_2) + \overline{RW_{in}R^{-1}(\overline{\bar{z}_1 + z_2})}) \qquad (1.4)$$

and for the out-going field

$$\phi_{out}(e_{z_1}, e_{z_2}) = \frac{1}{2}\exp(\langle z_1, z_2\rangle_{H^{1/2}(I\!R^3,\mathbb{C})})(RSR^{-1}(\bar{z}_1 + z_2) + \overline{RSR^{-1}(\overline{\bar{z}_1 + z_2})}), \qquad (1.5)$$

where R is the isomorphism,

$$R: \quad H^1(I\!R^3) \oplus L_2(I\!R^3) \to H^1(I\!R^3, \mathbb{C}),$$

and W_{in}, W_{out} are the in- and out-wave operator, respectively, of the classical nonlinear equation with the interaction of the 4th degree. This Wick symbol is defined for all coherent vectors with finite energy. Due to holomorphity the expression (1.4) is holomorphic in $\bar{z}_1, z_2$ for $z_1, z_2 \in H^1(I\!R^3, \mathbb{C})$, see [25].

The expression (1.4) gives, in fact, the mathematical realization of a physical idea to write an interacting field with the help of a wave operator. Due to holomorphity (1.4) has the quite correct sense written as

$$\phi = \frac{1}{2}\Big(: RW_{in}R^{-1}(\phi_{in}) : + : RW_{out}R^{-1}(\phi_{in}) :\Big),$$

i.e.

$$\phi = \sum_{n=1}^{\infty} \phi_n(: \phi_{in}...\phi_{in} :) = \sum_{n=1}^{\infty} \frac{1}{2}(R_n + \overline{R}_n)(: \phi_{in}...\phi_{in} :), \qquad (1.6)$$

where R_n is the unique generalized function from $\mathcal{S}(I\!R^{3n}, \mathbb{C})$, which corresponds to the n-th Frechét derivative of the function $RW_{in}R^{-1}$ at zero in the complex Hilbert space $H^1(I\!R^3, \mathbb{C})$ and $\overline{R}_n$ is its complex conjugation, i.e. (we denote later W_{in} as W without the subscript "in")

$$R_n(f_1 \otimes ... \otimes f_n) = \frac{1}{n!}d^n RWR^{-1}(0)(f_1, ..., f_n),$$

$f_j \in \mathcal{S}(I\!R^{3n}, \mathbb{C})$. In other words, R_n is equal to the n-th derivative of the function RWR^{-1} at zero and the derivative is taken over the positive frequency part of in-solution. The

positive frequency part [2] of *in*-solution may be defined with the help of the isomorphism R which gives the one-to-one correspondence with the initial *in*-data.

The expansion (1.6) is defined by the Taylor expansion of the (classical) wave operator at zero and it is convergent on coherent vectors near to the vacuum. The expansion (1.6) defines the bilinear form and the holomorphity allows to prove that the Wick kernel of (1.6) can be extended uniquely on all coherent vectors with finite energy (and is equal to (1.4)).

Thus, we construct the $:\phi_4^4:$ quantum field theory along the following pathway. To construct the $:\phi_4^4:$ quantum field theory we define the Wick kernel for the interacting quantum field as (1.4). As a consequence we obtain that the interacting quantum field satisfies the expansion (1.5) in the sense of bilinear forms on an appropriate subspace of the Fock space. The complex structure and estimates allow us to prove that (1.4) is defined correctly as a Wick kernel in the sense of Paneitz, Pedersen, Segal, and Zhou [34].

Separately, we prove that the solution of Eq. (1.3), defined as a bilinear form, has the Wick kernel equal to (1.4) on coherent vectors near to the vacuum, i.e. in a zero neighbourhood, [26], see also [22, 17, 36]. In addition, this Wick kernel (1.4) defines the unique operator–valued generalized function from $\mathcal{S}^1(I\!\!R^4) \times \mathcal{S}^1(I\!\!R^4)$ into $L(\Phi o\kappa)$. Here $\mathcal{S}^\alpha(I\!\!R^4)$ are the Gelfand spaces of test functions [14, v. 2], [15, v. 4], $L(\Phi o\kappa)$ is the space of linear bounded operators in the space $\Phi o\kappa$ and $\Phi o\kappa$ denotes the Fock space of the free quantum *in*-field. Moreover, this operator-valued generalized function can be extended on a Gelfand space $\mathcal{S}^\alpha$, $\alpha < 6/5$, and these Gelfand spaces with $\alpha > 1$ contain dense subspaces of functions with compact supports in coordinate space [28]. This result allows us to construct Wightman functions (as generalized function of Jaffe type) and matrix elements of the quantum scattering operator. It is possible to consider Wightman axioms: the conditions of positivity, spectrality, Poincaré invariance, locality, cluster property, and asymptotic completeness [29].

In this paper we introduce the Wick kernel (1.4) for the interacting quantum field of the $:\phi_4^4:$ quantum field theory and consider its properties.

2. Complex structure.

In this section we formulate and prove the assertions connected with the complex structure of solutions of the classical (real) nonlinear wave equation with the cubic nonlinearity in four–dimensional Minkowski space–time,

$$\Box u + m^2 u + \lambda u^3 = 0, \quad m > 0, \quad \lambda > 0. \tag{2.1}$$

We use these statements for the construction of the $:\phi_4^4:$ quantum field theory. To introduce the Wick kernel of the interacting quantum field of the $:\phi_4^4:$ theory we need the assertion about holomorphity (Theorem 2.1) and about T (and/or CPT) symmetry (Theorem 2.2).

[2] We use the usual "physical" terminology for this notion. It is defined uniquely for the massive linear Klein–Gordon equation, i.e. for free solutions.

The solution of Eq. (2.1) is uniquely determined by its Cauchy data

$$u|_{t=0} = \varphi, \quad \partial_t u|_{t=0} = \pi.$$

Let W_{in} $(= W)$ be the incoming wave operator (the *in*–wave operator, or the backward wave operator) of Eq. (2.1). The operator W_{in} maps *in*-data at time zero into initial data at time zero. Let W_{out} be the outgoing wave operator (the *out*-wave operator, or the forward wave operator). The operator W_{out} maps *out*–data into initial data. Let S be the scattering operator of the classical nonlinear wave equation (2.1).

Theorem 2.1 [25, Theorem 1.1].

Let $R(\varphi, \pi) = \varphi + i\mu^{-1}\pi$, $\mu = (-\Delta + m^2)^{1/2}$, be the invertible map of $\mathcal{S}_{Re}(I\!R^3) \oplus \mathcal{S}_{Re}(I\!R^3)$ onto $\mathcal{S}(I\!R^3, \mathbb{C})$. This map R defines also isomorphisms $(H^{1/2}(I\!R^3) \oplus H^{-1/2}(I\!R^3), J))$ onto $H^{1/2}(I\!R^3, \mathbb{C})$ and of $(H^1(I\!R^3) \oplus L_2(I\!R^3), J)$ onto $H^1(I\!R^3, \mathbb{C})$. Here

$$J = R^{-1} i R = \begin{pmatrix} 0 & -\mu^{-1} \\ \mu & 0 \end{pmatrix}.$$

The map RWR^{-1} is defined correctly as the mapping from $H^1(I\!R^3, \mathbb{C})$ onto $H^1(I\!R^3, \mathbb{C})$ and is the complex analytic mapping of the complex Hilbert space $H^1(I\!R^3, \mathbb{C})$ onto itself. In particular, for $z_{in}(\alpha) = \sum_{j=1}^{N} \alpha_j z_{in,j}$, $\alpha_j \in \mathbb{C}$, $z_{in,j} \in H^1(I\!R^3, \mathbb{C})$, $h \in H^{1/2}(I\!R^3, \mathbb{C})$, the functions $\langle h, RWR^{-1}(z_{in}(\alpha))\rangle_{H^{1/2}(I\!R^3, \mathbb{C})}$ are entire holomorphic function on $(\alpha_1, ...\alpha_N) \in \mathbb{C}^N$.

The same assertions are valid for the transformation $RW_{out}R^{-1}$ and RSR^{-1}.

Theorem 2.1 is proved in the paper [25, Theorem 1.1].

Let $\mathcal{P}$ be the Poincaré group. Let $\mathcal{P}_0$ be the connected component of the Poincaré group. Given $(a, \Lambda) \in \mathcal{P}$ and a finite–energy solution u of (2.1), define $(a, \Lambda)u$ by

$$((a, \Lambda)u)(t, x) = u((a, \Lambda)^{-1}(t, x)).$$

Then $(a, \Lambda)u$ is also a finite–energy solution of (2.1), and the map $((a, \Lambda), u) \to (a, \Lambda)u$ defines an action of the group $\mathcal{P}$ on the Hilbert space $H^1 \oplus L_2$ of initial data. We denote this map by $U(a, \Lambda)$. When the coupling constant in (2.1) $\lambda = 0$, i.e. for the case of the linear Klein–Gordon equation, the maps $U_0(a, \Lambda)$ are linear.

The wave operators have the following basic properties.

Theorem 2.2.

The wave operator W_{in} and W_{out} are correctly defined on the Hilbert space $H^1 \oplus L_2$. Moreover, they intertwine the free and interacting action of the Poincaré group, i.e.

$$U(a, \Lambda) = W_{in} U_0(a, \Lambda) W_{in}^{-1},$$

$$U(a, \Lambda) = W_{out} U_0(a, \Lambda) W_{out}^{-1}$$

for all $(a, \Lambda) \in \mathcal{P}_0$.

Proof of Theorem 2.2. See, for instance, [4]. Theorem 2.2 is proved.

Let Θ^T be the time reflection operator on the space of initial data, $\Theta^T(\varphi, \pi) = (\varphi, -\pi)$. Let Θ^P be the space reflection operator on the space of initial data, $\Theta^P(\varphi, \pi) = (\varphi^P, \pi^P)$, where $\varphi^P(x) = \varphi(-x)$, $\pi^P(x) = \pi^P(-x)$.

Theorem 2.3 (T-symmetry).
The equalities

$$W_{out} = \Theta^T W_{in} \Theta^T, \quad S^{-1} = \Theta^T S \Theta^T$$

are fulfilled.

Corollary 2.4 (PT- and CPT–symmetry).
The equalities

$$W_{out} = \Theta^T \Theta^P W_{in} \Theta^T \Theta^P, \quad S^{-1} = \Theta^T \Theta^P S \Theta^T \Theta^P$$

are fulfilled.

Remarks.

1. The symmetry Θ for the classical system means the following. Θ is a mapping of initial data, $(\varphi, \pi) \to \Theta(\varphi, \pi)$, if u is a solution of Eq. (2.1) with initial data (φ, π), then $\Theta(\varphi, \pi)$ are initial data of some solution u^Θ. In particular, if $u(t, x)$ is a solution of Eq. (2.1), then $u^{\Theta^T}(t, x) = u(-t, x)$ is the corresponding solution of Eq. (2.1).

2. Since P-symmetry (the space reflection symmetry) is obvious, so T-symmetry implies PT-symmetry and CPT-symmetry for the nonlinear equation (2.1). Of course, CPT-symmetry is more fundamental.

3. On the other hand Theorem 2.2 is connected with the correspondence between the time reflection and the complex conjugation. Theorem 2.2 appears in the fact that the interacting quantum field is a Hermitian bilinear form. For the bilinear form–solution this was pointed out in [17], [22].

4. The indexation of free solutions with the help of positive frequency part corresponds to the fact that the real diagonal of Wick symbol of the interacting quantum field satisfies the classical nonlinear wave equation.

Proof of Theorem 2.3.

Let $u(t, x)$ be a solution of classical nonlinear equation (2.1) with finite energy. Then $u(-t, x)$ (and $u(-t, -x)$) is a solution of the same equation also. The solution u defines the unique initial, *in*-, and *out*-data at time zero,

$$\begin{aligned} (\varphi(x), \pi(x)) &= (u(0, x), \dot{u}(0, x)), \\ (\varphi_{in}(x), \pi_{in}(x)) &= (u_{in}(0, x), \dot{u}_{in}(0, x)), \\ (\varphi_{out}(x), \pi_{out}(x)) &= (u_{out}(0, x), \dot{u}_{out}(0, x)), \end{aligned}$$

where u_{in} u_{out} are the unique free solutions such that

$$\|(u(t, \cdot), \dot{u}(t, \cdot)) - (u_{in}(t, \cdot), \dot{u}_{in}(t, \cdot))\|_{H^1 \oplus L_2} \quad \text{for} \quad t \to -\infty$$

and

$$\|(u(t,\cdot),\dot u(t,\cdot))-(u_{out}(t,\cdot),\dot u_{out}(t,\cdot))\|_{H^1\oplus L_2} \quad \text{for} \quad t\to+\infty.$$

This implies immediately that the solution u^{Θ^T} $(=u(-t,x))$ has the initial data equal to $(\varphi,-\pi)$ at time $t=0$, the *in*–data equal to $(\varphi_{out},-\pi_{out})$, and the *out*-data equal to $(\varphi_{in},-\pi_{in})$.

Therefore, since by definition the wave operators satisfy the following equalities

$$W_{in}(\varphi_{in},\pi_{in})=(\varphi,\pi), \quad W_{out}(\varphi_{out},\pi_{out})=(\varphi,\pi),$$

$$S(\varphi_{in},\pi_{in})=(\varphi_{out},\pi_{out}), \quad S=W_{out}^{-1}W_{in},$$

the previous assertion implies the equality

$$\Theta^T W_{in}(\varphi_{in},\pi_{in})=W_{out}\Theta^T(\varphi_{in},\pi_{in}),$$

i.e.

$$W_{out}=\Theta^T W_{in}\Theta^T.$$

The last equality implies that

$$S^{-1}=\Theta^T S\Theta^T.$$

Theorem 2.3 is proved.

Remark.
It is well known that S is not the identity operator [20]. This implies that $W_{out}\neq W_{in}$.

3. Wick kernel of the interacting quantum field.

The existence of classical wave operator and holomorphity allow us to introduce the Wick kernel of the interacting quantum field and to construct the bilinear form given by this kernel.

For this purpose we introduce the Hilbert Fock space for the free quantum field ϕ_{in}. We introduce the Fock space as it described in [36], but instead of the basis used by Rączka [36] we take a basis in the complex Hilbert space $H^{1/2}(I\!R^3,\mathbb{C})$ with explicit introduction of its pure real and pure imaginary parts (for the integral by Paneitz, Pedersen, Segal, Zhou [34], see also [5], we may use any basis in the complex Hilbert space).

Let us take a (canonical and standard) basis in the space $H^{1/2}(I\!R^3,\mathbb{C})$ analogous to the basis introduced in [7, ch. 5.1, p. 141] (the space $H^{1/2}(I\!R^3,\mathbb{C})$ corresponds to the dual space with dualization $(.,.)_{L_2}$ of the one-particle subspace of the Fock space, this one-particle subspace is denoted by Bogoliubov, Logunov, Todorov [7, ch. 5.1, p. 141] as $\mathcal{H}_1$).

Similar to [7, ch.5.1, p. 141] we introduce our basis and note that all basic vectors in our basis are pure real–valued functions in coordinate space. Thus, we define for the (relativistic) coordinate operator (in momentum space)

$$\mathbf{q}=\sqrt{\mu(p)}\,i\nabla_{\mathbf{p}}\frac{1}{\sqrt{\mu(p)}}=i\nabla_{\mathbf{p}}-\frac{i}{2}\frac{\mathbf{p}}{m^2+\mathbf{p}^2},$$

in this case the momentum operator is the operator of multiplication on $\mathbf{p}$ (in momentum space).

The operators q^j and p^j, $j = 1, 2, 3$, satisfy the canonical commutation relations

$$[q^j, p^{j'}] = i\delta_{j,j'}, \quad [q^j, q^{j'}] = [p^j, p^{j'}] = 0,$$

are hermitian (and essentially self–adjoint on $\mathcal{S}(I\!R^3, \mathbb{C})$) in the Hilbert space $\mathcal{H}_1$ with the scalar product

$$(\psi_1, \psi_2) = \int \overline{\psi_1(p)}\psi_2(p)\frac{d^3p}{\mu(p)},$$

the Hilbert space $\mathcal{H}_1$ coincides with the Hilbert space $H^{-1/2}(I\!R^3, \mathbb{C})$ in coordinate space.

We define the operators

$$b_j = \frac{1}{\sqrt{2}}(q^j + ip^j), \quad b_j^* = \frac{1}{\sqrt{2}}(q^j - ip^j), \quad j = 1, 2, 3.$$

These operators are mutually adjoint and correspond to the analogous operators in [7, ch. 5.1, p. 141],

$$b_j = ib_j^{BLT} \quad ([7, \text{ch. } 5.1, \text{p.}141]),$$
$$b_j^* = -ib_j^{BLT} \quad ([7, \text{ch. } 5.1, \text{p. } 141]).$$

They satisfy the commutation relations

$$[b_j, b_{j'}^*] = \delta_{j,j'}, \quad [b_j, b_{j'}] = [b_j^*, b_{j'}^*] = 0.$$

Let e_0 be the normalized vector, satisfying the equalities

$$b_j\ e_0 = 0, \quad j = 1, 2, 3.$$

In the considered momentum space

$$e_0(p) = \pi^{-3/4}(m^2 + p^2)^{1/4}e^{-p^2/2}.$$

The orthonormalized basic vectors in $\mathcal{H}_1$ we define as the Hermite functions (the Hermite polynomials) defined on the mass hyperboloid,

$$e_k \equiv e_{k_1k_2k_3} = \frac{(b_1^*)^{k_1}(b_2^*)^{k_2}(b_3^*)^{k_3}}{\sqrt{k_1!k_2!k_3!}}e_0, \quad k_1, k_2, k_3 = 0, 1, \ldots .$$

In x–space the vector $e_0(x)$ is a real–valued function and the vectors $e_k(x)$ are real–valued functions also. This follows from the fact, that in the p–space

$$b_j^* = \frac{1}{\sqrt{2}}(\mu^{1/2}i\nabla_{p^j}\mu^{-1/2} - ip^j) = \frac{1}{\sqrt{2}}(\mu^{1/2}x^j\mu^{-1/2} - \frac{\partial}{\partial x^j})$$

and from the reality of operators μ as convolution operators. Here

$$q = \mu^{1/2}x\mu^{-1/2} = \mu^{1/2}i\nabla_p\mu^{-1/2}$$

is the operator of relativistic coordinate in p–space (= in momentum space), see [7, ch. 5.1, (2.5.6), p. 140], $\mu^{-1/2} = (-\Delta + m^2)^{-1/2}$ and gives an isomorphism $L^2(\mathbb{R}^3) \to H^{1/2}(\mathbb{R}^3)$.

In x-space the coordinate operator has the form

$$q = \mu^{-1/2} x \mu^{1/2}$$

because the function $F(x)$ from the one-particle space of the Fock space corresponds to the function

$$\mu(\mathcal{F}F)(p)$$

in the p–space, where $\mathcal{F}$ is the Fourier transform. If, correspondingly, $F(p)$ is a function that is square integrable over $d^3p/\mu(p)$, then in x–space this function corresponds to the function $\mu^{-1}(\mathcal{F}F)(x)$ and the operator of relativistic coordinate is $\mu^{-1/2} x^j \mu^{1/2}$ (here all identifications appear as the consideration of the same solution in different indexing spaces).

In this case, all functions that define the basis are orthonormal and real–valued in the x–space. In the p–space these functions satisfy the following relation

$$F(p) = \overline{F(-p)}.$$

The functions

$$\varphi_k(x) = c \int \exp(-ipx) e_k(p) d\rho(p),$$

$$c = [2(2\pi)^3]^{-1/2}, \quad d\rho(p) = d^3p/\mu(p), \quad \mu(p) = (p^2 + m^2)^{1/2},$$

are orthonormal real in the complex Hilbert space $H^{1/2}(\mathbb{R}^3, \mathbb{C})$ and generate a basis in this Hilbert space.

We use the following representation of the free (scalar, hermitian) quantum field in the Fock space, see [5], [7, ch. 5.1], [36]. We denote the Fock space for the free field as $\Phi o\kappa$ and chose it as

$$\Phi o\kappa = \oplus_{n=0}^{\infty} \mathcal{H}_n,$$

where $\mathcal{H}_0 = \mathbb{C}$, $\mathcal{H}_1 = H^{-1/2}(\mathbb{R}^3, \mathbb{C})$, $\mathcal{H}_n = \mathrm{sym}\ \widehat{\otimes}_n H^{-1/2}(\mathbb{R}^3, \mathbb{C})$. The Fock space $\Phi o\kappa$ is associated with the free quantum field ϕ_{in}. Let $\phi_{in}^-(0,x)$, $\phi_{in}^+(0,x)$ be the annihilation and creation part of the free quantum field, i.e. the negative and positive frequency part of the field ϕ_{in} at time zero. Using the chosen basis $\{e_k(p)\}_{k=0}^{\infty}$ we define the smeared annihilation and creation operators a_k and a_k^*,

$$a_k = \int \phi_{in}^-(0,x) \tilde{e_k}(x) d^3x, \quad a_k^* = \int \phi_{in}^+(0,x) \tilde{e_k}(x) d^3x,$$

where $\tilde{e_k}(x)$ is a basis vector in coordinate space (i.e. the Fourier transform of the vector $e_k(p)$).

The operators a_k, a_k^*, satisfy the canonical commutation relations $[a_k, a_l^*] = \delta_{kl}$ and form an infinite-dimensional nilpotent Lie algebra, which is irreducible in the Fock space $\Phi o\kappa$.

The free quantum field $\phi_{in}(t,x)$ has the following form in terms of annihilation a_k and creation a_k^* operators:

$$\phi_{in}(t,x) = \sum_k (\varphi_k(t,x)a_k^* + \overline{\varphi_k(t,x)}a_k),$$

where

$$\varphi_k(t,x) = c\int \exp(i\mu(p)t - ipx)e_k(p)d\rho(p),$$

$$d\rho(p) = d^3p/\mu(p), \quad \mu(p) = (p^2+m^2)^{1/2}, \quad c = [2(2\pi)^3]^{-1/2},$$

$$\varphi_k(0,x) \equiv \varphi_k(x) = c\int \exp(-ipx)e_k(p)d\rho(p),$$

$$\varphi_k(x) = \overline{\varphi_k(x)},$$

with our choice of the real basis, $e_k(p) = \overline{e_k(-p)}$. In this case

$$\overline{\varphi_k(t,x)} = \varphi_k(-t,x).$$

The vacuum averages of a sum of products of the field ϕ_{in} coincide with the corresponding value of the free Wightman functional.

Now we construct convenient dense subspaces contained in the Fock space. Let

$$e_z \equiv e(z) = \exp(za^*)\Omega, \tag{3.1}$$

$$|z\rangle = \exp(-\frac{1}{2}\|z\|^2)\exp(za^*)\Omega,$$

where

$$\|z\|^2 = \sum_{k=0}^{\infty}|z_k|^2, \quad za^* = \sum_{k=0}^{\infty} z_k a_k^*,$$

z is an element of the complex Hilbert space $H^{1/2}(I\!R^3, \mathbb{C})$, $z_k = \langle\varphi_k, z\rangle_{H^{1/2}(I\!R^3,\mathbb{C})}$. The vectors e_z, $z \in H^{1/2}(I\!R^3,\mathbb{C})$, are called the coherent vectors and the vectors $|z\rangle$ are called the coherent state vectors. It follows from Eq. (3.1) that

$$(\exp(z'a^*)\Omega, \exp(za^*)\Omega) = \exp(\langle z', z\rangle_{H^{1/2}(I\!R^3,\mathbb{C})}) = \exp(\sum_{k=0}^{\infty}\bar{z}_k' z_k)$$

and

$$\begin{aligned}\langle z'|z\rangle &= \exp\{-\sum_{k=0}^{\infty}[\frac{1}{2}|z_k' - z_k|^2 - i\ \mathrm{Im}\ (\bar{z}_k' z_k)]\} \\ &= \exp(\sum_{k=0}^{\infty}(\bar{z}_k' z_k - \frac{1}{2}|z_k'|^2 - \frac{1}{2}|z_k|^2))\end{aligned}$$

in the chosen earlier basis

$$z = \sum z_k\varphi_k, \quad z_k = \langle\varphi_k, z\rangle_{H^{1/2}(I\!R^3,\mathbb{C})},$$

$$\langle z_1, z_2\rangle_{H^{1/2}(I\!R^3,\mathbb{C})} = \sum_{k=0}^{\infty} \bar{z}_{1,k} z_{2,k}.$$

Let A be a subset in $H^{1/2}(I\!R^3,\mathbb{C})$. Let $D_{coh}(A)$ be the subspace in the Fock space of the *in*-field generated by finite linear combinations of coherent vectors from A, i.e. if $\chi \in D_{coh}(A)$, then

$$D_{coh}(A) = \{\chi \in D_{coh}(A) \,|\, \chi = \sum \alpha_j e(z_j),\ \textit{the sum is finite}, \alpha_j \in \mathbb{C}, z_j \in A\}.$$

We define the Wick kernel for the interacting quantum field as the function $(z_1, z_2) \to \phi(e_{z_1}, e_{z_2})$ from $H^1(I\!R^3,\mathbb{C}) \times H^1(I\!R^3,\mathbb{C})$ into $H^1(I\!R^3,\mathbb{C})$,

$$\phi(e_{z_1}, e_{z_2}) = \exp(\langle z_1, z_2\rangle_{H^{1/2}(I\!R^3,\mathbb{C})})\frac{1}{2}(RWR^{-1}(\bar{z}_1 + z_2) + \overline{RWR^{-1}(\overline{\bar{z}_1 + z_2})}),$$

where RWR^{-1} maps $H^1(I\!R^3,\mathbb{C})$ into $H^1(I\!R^3,\mathbb{C})$. Thus, the Wick kernel smoothed with some test function over space is equal to

$$\int (\phi(e_{z_1}, e_{z_2}))(x)h(x)d^3x = \exp(\langle z_1, z_2\rangle)$$

$$\frac{1}{2}\Big(\int (\text{ Re } RWR^{-1}(\bar{z}_1 + z_2))(x)h(x)d^3x + i\int (\text{ Im } RWR^{-1}(\bar{z}_1 + z_2))(x)h(x)d^3x$$
$$+\int (\text{ Re } RWR^{-1}(\overline{\bar{z}_1 + z_2}))(x)h(x)d^3x - i\int (\text{ Im } RWR^{-1}(\overline{\bar{z}_1 + z_2}))(x)h(x)d^3x\Big)$$
$$= \exp(\langle z_1, z_2\rangle)\frac{1}{2}\Big(\langle \overline{RWR^{-1}(\bar{z}_1 + z_2)}, \mu^{-1}h\rangle_{H^{1/2}(I\!R^3,\mathbb{C})} + \langle RWR^{-1}(\overline{\bar{z}_1 + z_2}), \mu^{-1}h\rangle_{H^{1/2}(I\!R^3,\mathbb{C})}\Big).$$

The smoothed Wick kernel corresponds to the smoothed field at the time zero, i.e. to the bilinear form corresponding to $\int \phi(0,x)h(x)d^3x$.

Analogously, we define the Wick kernel for the *out*-going quantum field as

$$\phi_{out}(e_{z_1}, e_{z_2}) = \exp(\langle z_1, z_2\rangle_{H^{1/2}(I\!R^3,\mathbb{C})})\frac{1}{2}(RSR^{-1}(\bar{z}_1 + z_2) + \overline{RSR^{-1}(\overline{\bar{z}_1 + z_2})}).$$

In [26] we show that the interacting and *out*-going Wick kernel can be reconstructed uniquely with the help of the bilinear form–solution, that has been constructed in [22]. In this paper we consider the introduced Wick kernels, their properties and the bilinear form defined by these Wick kernels.

Theorem 3.1.
The Wick kernel

$$\phi(e_{z_1}, e_{z_2}) = \exp(\langle z_1, z_2\rangle_{H^{1/2}(I\!R^3,\mathbb{C})})\frac{1}{2}(RWR^{-1}(\bar{z}_1 + z_2) + \overline{RWR^{-1}(\overline{\bar{z}_1 + z_2})})$$

is correctly defined on $H^1(I\!R^3,\mathbb{C})$ $(\subset H^{1/2}(I\!R^3,\mathbb{C}))$ *as the map from*

$$H^1(I\!R^3,\mathbb{C}) \times H^1(I\!R^3,\mathbb{C})$$

into $H^1(\mathbb{R}^3,\mathbb{C})$ and is complex antiholomorphic on z_1 and complex holomorphic on z_2. In particular, for z_1, z_2 belonging to finite–dimensional subspaces in $H^1(\mathbb{R}^3,\mathbb{C})$,

$$z_1(\alpha_1)=\sum_{j=1}^{n}\alpha_{1,j}z_{1,j},\quad z_2(\alpha_2)=\sum_{j=1}^{n}\alpha_{2,j}z_{2,j},\quad z_{1,j},z_{2,j}\in H^1(\mathbb{R}^3,\mathbb{C}),\ \ \alpha_{1,j},\alpha_{2,j}\in\mathbb{C},$$

$h\in H^1(\mathbb{R}^3,\mathbb{C})$, the function

$$\langle h,\phi(e(z_1(\alpha_1)),e(z_2(\alpha_2)))\rangle_{H^{1/2}(\mathbb{R}^3,\mathbb{C})}$$

is an entire complex antiholomorphic function on $\alpha_1\in\mathbb{C}^n$ and an entire complex holomorphic function on $\alpha_2\in\mathbb{C}^n$ in the usual sense. Furthermore,

$$\overline{\langle h,\phi(e_{z_1},e_{z_2})\rangle}_{H^{1/2}(\mathbb{R}^3,\mathbb{C})}=\langle\bar h,\phi(e_{z_2},e_{z_1})\rangle_{H^{1/2}(\mathbb{R}^3,\mathbb{C})},$$

i.e.

$$\overline{\phi(e_{z_1},e_{z_2})}=\phi(e_{z_2},e_{z_1}),$$

where a complex conjugation is defined as the complex conjugation of a function with values in $H^1(\mathbb{R}^3,\mathbb{C})$. It is valid the following estimate

$$|\langle h,\phi(e_{z_1},e_{z_2}\rangle_{H^{1/2}(\mathbb{R}^3,\mathbb{C})}|\le c\|h\|_{L_2(\mathbb{R}^3,\mathbb{C})}\exp(Re(\langle z_1,z_2\rangle_{H^{1/2}(\mathbb{R}^3,\mathbb{C})}))\ \|\bar z_1+z_2\|_{H^1(\mathbb{R}^3,\mathbb{C})}.$$

The analogous assertion is also valid for the out-going field ϕ_{out}.

The proof of Theorem 3.1 follows from the assertion of Theorem 2.1.

Remark.

The Hermitian symmetry is implied by the explicit expression of the Wick kernel and by the equality

$$W_{in}=\Theta^T W_{out}\Theta^T,$$

or

$$\overline{RW_{in}R^{-1}z}=R\Theta^T W_{in}\Theta^T R^{-1}\bar z=RW_{out}R^{-1}\bar z$$

for the wave operators W_{in}, W_{out}. In particular, for any real test function h

$$\int\phi(e_{z_1},e_{z_2})(0,x)h(x)d^3x=\int\overline{\phi(e_{z_2},e_{z_1})}(0,x)h(x)d^3x,$$

the integral is defined correctly (because $\phi(e_{z_1},e_{z_2})\in H^1(\mathbb{R}^3,\mathbb{C})$ for $z_1,z_2\in H^1(\mathbb{R}^3,\mathbb{C})$).

This Wick kernel defines a bilinear form on $D_{coh}(H^1(\mathbb{R}^3,\mathbb{C}))\times D_{coh}(H^1(\mathbb{R}^3,\mathbb{C}))$, i.e. on the subspace in the Fock space of the *in*–field generated by finite linear combinations of coherent vectors with finite energy.

Theorem 3.2.

Let $\chi_1,\chi_2\in D_{coh}(H^1(\mathbb{R}^3,\mathbb{C}))$, $\chi_1=\sum\alpha_{1,j}e_{z_{1,j}}$, $\chi_2=\sum\alpha_{2,j}e_{z_{2,j}}$, then

$$\phi(\chi_1,\chi_2)=\sum\overline{\alpha}_{1,j}\alpha_{2,j}\phi(e_{z_{1,j}},e_{z_{2,j}})\tag{3.2}$$

is a bilinear form with values in $H^1(\mathbb{R}^3,\mathbb{C})$. *In addition, the expression*

$$\langle h,\phi(\chi_1,\chi_2)\rangle_{H^{1/2}(\mathbb{R}^3,\mathbb{C})}=\sum\overline{\alpha}_{1,j}\alpha_{2,j}\langle h,\phi(e_{z_{1,j}},e_{z_{2,j}})\rangle_{H^{1/2}(\mathbb{R}^3,\mathbb{C})}$$

defines the bilinear form also. The bilinear forms $\phi(\chi_1,\chi_2)$ *and* $\langle h,\phi(\chi_1,\chi_2)\rangle_{H^{1/2}(\mathbb{R}^3,\mathbb{C})}$ *are Hermitian symmetric*

$$\overline{\phi(\chi_1,\chi_2)}=\phi(\chi_1,\chi_2),$$
$$\overline{\langle h,\phi(\chi_1,\chi_2)\rangle}_{H^{1/2}(\mathbb{R}^3,\mathbb{C})}=\langle\overline{h},\phi(\chi_1,\chi_2)\rangle_{H^{1/2}(\mathbb{R}^3,\mathbb{C})}.$$

The analogous assertion is also valid for the out-going quantum field ϕ_{out}.

Let H_{in} be the Hamiltonian of the (free quantum) *in*-coming field ϕ_{in}. Due to the energy conservation it is equal to the total Hamiltonian.

Theorem 3.3 (Energy positivity and holomorphity).
Let $\chi_1=\sum\alpha_{1,j}e_{z_{1,j}}$, $\chi_2=\sum\alpha_{2,j}e_{z_{2,j}}$, $\chi_1,\chi_2\in D_{coh}(H^1(\mathbb{R}^3,\mathbb{C}))$. *The expression*

$$\phi(\exp(it_1H_{in}-s_1H_{in})\chi_1,\exp(it_2H_{in}-s_2H_{in})\chi_2)$$
$$=\sum\overline{\alpha}_{1,j}\alpha_{2,j}\phi(e(\exp(i\mu t_1-\mu s_1)z_{1,j}),e(\exp(i\mu t_2-\mu s_2)z_{2,j}))\qquad(3.3)$$

is correctly defined for $Im\ s_1\geq 0$, $Im\ s_2\geq 0$ *as a bilinear form on*

$$D_{coh}(H^1(\mathbb{R}^3,\mathbb{C}))\times D_{coh}(H^1(\mathbb{R}^3,\mathbb{C}))$$

with values in $H^1(\mathbb{R}^3,\mathbb{C})$. *Expression (3.3) depends antiholomorphically on* t_1+is_1 *and holomorphically on* t_2+is_2).

The analogous assertion is also valid for the out-going quantum field ϕ_{out}.

Proof of Theorem 3.2 and 3.3. It is evident that Theorem 3.2 is the consequence of Theorem 3.3 with $t_1=t_2=s_1=s_2=0$. Therefore, it is sufficient to prove Theorem 3.3.

The assertion about holomorphity on $\overline{t_1+is_1}$, t_2+is_2 is implied by Theorem 3.1 (or by Theorem 2.1) and by holomorphity of the operator $\exp(i\mu t-\mu s)=\exp i\mu(t+is)$ on $t+is$ for $s>0$.

Now we prove that the expression ϕ is a correctly defined bilinear form. We consider here two variants of the proof for the interacting field ϕ. The field ϕ_{out} can be considered analogously.

The first variant of the proof uses Theorem 1 [34, Theorem 1]. The Wick kernel is given by nonlinear operators RWR^{-1} and $R\Theta^TW\Theta^TR^{-1}$. The complex holomorphity of the considered Wick kernel was proved in [25]. In order to use Theorem 1 [34, Theorem 1] we take as an operator B the operator

$$B=\mu+\mu^{-1/2}(1+x^2)\mu^{1/2}.$$

It is evident, that this operator is positive in the coordinate space $H^{1/2}(\mathbb{R}^3,\mathbb{C})$, self-adjoint, and $\exp(-sB)$ is a trace class operator for $s>0$.

Remark.

It is possible to use more simple choice of (positive, self-adjoint) operator B, for instance,

$$B = \mu^{\alpha} + \mu^{-1/2}(1+x^2)^{\beta}\mu^{1/2},$$

or

$$B = \mu^{\alpha-1/2}(1+x^2)^{\beta}\mu^{\alpha+1/2},$$

for sufficiently large α, β, such that the operator B^{-1} is nuclear (as a product of two Hilbert-Schmidt operators).

We now consider the Wick kernel $\langle h, \phi(e(e^{-s_1 B}z_1), e(e^{-s_2 B}z_2)\rangle$. It is easily to see that this kernel satisfies the estimate (2) of Theorem 1 [34, Theorem 1]. This is implied by the explicit form of the Wick kernel, conservation of energy (which gives an estimate for the nonlinear terms), and by a simple estimate $B \le c(s)\exp(sB)$, $s > 0$. Therefore, the conditions of Theorem 1 [34, Theorem 1] are fulfilled and ϕ is a correctly defined bilinear form. It is possible to extend this bilinear form by continuity on the subspace generated by finite linear combinations of coherent vectors with finite energy.

The first proof of Theorem 3.3 and Theorem 3.2 is complete.

The second proof of Theorem 3.3. Let $\chi_1(z)$, $\chi_2(z)$ be the holomorphic functions on the space $H^1(I\!R^3, \mathbb{C})$ ($\subset H^{1/2}(I\!R^3, \mathbb{C})$) corresponding to the vectors χ_1, χ_2 in the complex wave representation of the Fock space (see [5, ch. 1, Theorem 1.13, p. 67-68]. Let θ be strictly positive, $\theta > 0$, and such that for $\beta_1, \beta_2 \in \mathbb{C}$, $|\beta_1| + |\beta_2| < \theta$, the Taylor series in β_1, β_2 for the functions $\phi(e(\beta_1 z_{1,j_1}), e(\beta_2 z_{2,j_2}))$ converge in $H^1(I\!R^3, \mathbb{C})$ for all j_1, j_2 (χ_1, χ_2 are finite linear combinations of coherent vectors). Let

$$\chi_1(\beta_1) = \chi_1(\beta_1 z), \quad \chi_2(\beta_2) = \chi_2(\beta_2 z).$$

For our sufficiently small β_1, β_2, $|\beta_1| + |\beta_2| < \theta$, we write the following equality

$$\sum_{j_1,j_2} \overline{\alpha}_{1,j_1}\alpha_{2,j_2}\phi(e(\beta_1 z_{1,j_1}), e(\beta_2 z_{2,j_2}))$$

$$= \sum_{j_1,j_2} \overline{\alpha}_{1,j_1}\alpha_{2,j_2}\exp(\langle \beta_1 z_{1,j_1}, \beta_2 z_{2,j_2}\rangle) \sum_{n=1}^{\infty} \phi_n(\underbrace{(\overline{\beta}_1\overline{z}_{1,j_1} + \beta_2 z_{2,j_2}) \otimes ... \otimes (\overline{\beta}_1\overline{z}_{1,j_1} + \beta_2 z_{2,j_2})}_{n})$$

$$= \lim_{N\to\infty}\lim_{V,\sigma}\sum_{n=1}^{N}\sum_{j_1,j_2} \overline{\alpha}_{1,j_1}\alpha_{2,j_2}\exp(\langle \beta_1 z_{1,j_1}, \beta_2 z_{2,j_2}\rangle)$$
$$\phi_n(\underbrace{(\overline{\beta}_1\overline{z}_{1,j_1,V,\sigma} + \beta_2 z_{2,j_2,V,\sigma}) \otimes ... \otimes (\overline{\beta}_1\overline{z}_{1,j_1,V,\sigma} + \beta_2 z_{2,j_2,V,\sigma})}_{n}). \tag{3.4}$$

Here σ is an ultraviolet cut-off and V is a space cut-off (i.e. σ is the smoothing with some test function and V is the multiplication on some test function, σ tends to the δ-function and V tends to 1). Further,

$$\phi_n = \frac{1}{2}(R_n + \overline{R}_n)$$

is a tempered distribution (i.e. it is a generalized function from the Schwartz space), R_n is the tempered distribution defined uniquely by the n-linear continuous form

$$\frac{1}{n!} d^n RWR^{-1}(0)$$

on $H^1(I\!R^3, \mathbb{C}) \otimes ... \otimes H^1(I\!R^3, \mathbb{C})$ and by the Schwartz nuclear theorem. We denote this n-linear form and the generalized function by the same notation R_n. The generalized function R_n is such that

$$R_n(f_1 \otimes ... \otimes f_n) = \frac{1}{n!} d^n RWR^{-1}(0)(f_1, ..., f_n).$$

A Wick polynomial is a correctly defined bilinear form, for instance, on

$$D_{coh}(\mathcal{S}(I\!R^3, \mathbb{C})) \times D_{coh}(\mathcal{S}(I\!R^3, \mathbb{C})).$$

This is implied easily by the explicit form of Wick monomial

$$: \phi_{in}(f_1)...\phi_{in}(f_n) : (\chi_1, \chi_2) = 2^{-n/2} \sum_{K \subseteq \{1,...,n\}} \langle \prod_{k \in K} a(f_k)\chi_1, \prod_{k \in \{1,...,n\} \setminus K} a(f_k)\chi_2 \rangle,$$

where a is an annihilation operator. This explicit form allows to extend these expressions by continuity and linearity on tempered generalized functions, see, for instance, Reed, Simon [39, v. 2], [3, Theorem 3 with appropriate operator A].

Therefore, the expressions

$$\phi_n(: \phi_{in}...\phi_{in} :),$$

$\phi_n \in \mathcal{S}'(I\!R^{3n}, H^1(I\!R^3, \mathbb{C}))$, is correctly defined as a bilinear form on

$$D_{coh}(\mathcal{S}(I\!R^3, \mathbb{C})) \times D_{coh}(\mathcal{S}(I\!R^3, \mathbb{C}))$$

with values in $H^1(I\!R^3, \mathbb{C})$. On vectors $\chi_1, \chi_2 \in D_{coh}(\mathcal{S}(I\!R^3, \mathbb{C}))$ this bilinear form is equal to

$$\phi_n(: \phi_{in}...\phi_{in} :)(\chi_1, \chi_2)$$
$$= \sum_{j_1, j_2} \overline{\alpha}_{1,j_1} \alpha_{2,j_2} \exp(\langle z_{1,j_1}, z_{2,j_2} \rangle_{H^{1/2}(I\!R^3, \mathbb{C})}) \phi_n((\overline{z}_{1,j_1} + z_{2,j_2}) \otimes ... \otimes (\overline{z}_{1,j_1} + z_{2,j_2})).$$

Now if $\chi_1, \chi_1', \chi_2, \chi_2', \in D_{coh}(\mathcal{S}(I\!R^3, \mathbb{C}))$,

$$\chi_1 = \chi_1', \quad \chi_2 = \chi_2',$$

we define $\chi_1(\beta_1), \chi_1'(\beta_1), \chi_2(\beta_2), \chi_2'(\beta_2)$. It is clear that

$$\chi_1(\beta_1) = \chi_1'(\beta_1), \quad \chi_2(\beta_2) = \chi_2'(\beta_2).$$

Thus, for sufficiently small θ, $|\beta_1| + |\beta_2| < \theta$, Eq. (3.4) implies that

$$\phi(\chi_1(\beta_1), \chi_2(\beta_2)) = \lim_N \lim_{V,\sigma} \sum_{n=1}^{N} \phi_n((\underbrace{: \phi_{in}...\phi_{in} :}_{n})(\chi_1(\beta_1), \chi_2(\beta_2))$$

$$= \lim_N \lim_{V,\sigma} \sum_{n=1}^{N} \phi_n((\underbrace{:\phi_{in}...\phi_{in}:}_{n})(\chi'_1(\beta_1), \chi'_2(\beta_2)) = \phi(\chi'_1(\beta_1), \chi'_2(\beta_2))$$

i.e.

$$\phi(\chi_1(\beta_1), \chi_2(\beta_2)) = \phi(\chi'_1(\beta_1), \chi'_2(\beta_2)) \tag{3.5}$$

for small β_1, β_2. Since $\phi(\chi_1(\beta_1), \chi_2(\beta_2))$ and $\phi(\chi'_1(\beta_1), \chi'_2(\beta_2))$ are entire holomorphic functions on $\bar{\beta}_1$, β_2 Eq. (3.5) for small β_1, β_2 implies that it is fulfilled for all $\beta_1, \beta_2 \in \mathbb{C}$, in particular, for $\beta_1 = \beta_2 = 1$, that is,

$$\phi(\chi_1, \chi_2) = \phi(\chi_1(1), \chi_2(1)) = \phi(\chi'_1(1), \chi'_2(1)) = \phi(\chi'_1, \chi'_2).$$

The proof of linearity in χ_1, χ_2 is analogous.

Therefore, (3.2) and (3.3) are correctly defined bilinear forms. The second proof of Theorem 3.2 and Theorem 3.3 is complete.

Theorem 3.4 (Poincaré invariance).
The Wick kernel satisfies the following equalities

$$\phi(e_{z_1}, e_{z_2})((a,\Lambda)^{-1}(t,x)) = \phi(e(RU_0(a,\Lambda)R^{-1}z_1), e(RU_0(a,\Lambda)R^{-1}z_2))(t,x),$$

$$\phi_{out}(e_{z_1}, e_{z_2})((a,\Lambda)^{-1}(t,x)) = \phi_{out}(e(RU_0(a,\Lambda)R^{-1}z_1), e(RU_0(a,\Lambda)R^{-1}z_2))(t,x).$$

Here $(a,\Lambda) \in \mathcal{P}_0$,

$$\phi(e_{z_1}, e_{z_2})((t,x)) = \phi(e(RU_0(-t)R^{-1}z_1), e(RU_0(-t)R^{-1}z_2))(0,x),$$

$$\phi_{out}(e_{z_1}, e_{z_2})((t,x)) = \phi_{out}(e(RU_0(-t)R^{-1}z_1), e(RU_0(-t)R^{-1}z_2))(0,x),$$

$U_0(t) = U_0(a,\Lambda)|_{(a,\Lambda)=((t,0),1)}$.

Proof of Theorem 3.4. Theorem 3.4 is the direct consequence of Theorem 2.2. Theorem 3.4 is proved.

4. Conclusion.

In this paper with the help of the complex structure of the classical nonlinear wave equation we introduce the Wick kernel of the interacting quantum field. In the next paper [28] we prove that the constructed Wick kernel defines the operator-valued generalized function from a Gelfand space $\mathcal{S}^\alpha(\mathbb{R}^4)$, $\alpha > 1$ [14]. The proof uses mainly the technique of works mentioned in the references.

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References

[1] Aizenman, M.: Geometric analysis of ϕ^4 fields and Ising models (Part I & II). Commun. Math. Phys. **86** (1982) 1-48.

[2] Aizenman, M., Graham, R.: On the renormalized coupling constant and the susceptibility in ϕ_4^4 field theory and the Ising model in four dimensions. Nucl. Phys. **B225** [**FS9**] (1983) 261-288.

[3] Baez, J.C.: Wick products of the free Bose field. J. Funct. Anal. **86** (1989) 211-225.

[4] Baez, J.: Scattering and complete integrability in four dimensions. Mathematical aspects of classical field theory. Proceedings of the AMS-IMS-SIAM. Joint Summer Research Conference held July 20-26, 1991. Ed. M.J.Gotay, J.E.Marsden, V.Moncrief. In: Contemporary Mathematics, **v. 132**. Rhode Island, AMS Providence, 1992, pp. 99-116.

[5] Baez, J., Segal, I.E., Zhou, Z.: Introduction to algebraic quantum field theory. Princeton University Press, 1992.

[6] Berezin, F.A.: The method of second quantization. New York, Academic Press, 1966.

[7] Bogoliubov, N.N., Logunov, A.A., Todorov, I.T.: The foundations of axiomatic approach to quantum field theory. Moscow, Nauka, 1969.

[8] Bogoliubov, N.N., Logunov, A.A., Oksak, A.I., Todorov, I.T.: The basic principles of quantum field theory. Moscow, Nauka, 1987.

[9] Bogoliubov, N.N., Medvedev, B.V., Polivanov, M.K.: Voprosi teorii dispersionnih sootnoshenij. Moscow, Fizmatgiz, 1958.

[10] Bogoliubov, N.N., Shirkov, D.V.: Introduction to the theory of quantized fields. New York, Interscience, 1959.

[11] Callaway, D.: Triviality pursuit: can elementary scalar particles exist? Phys. Reports **167** No 5 (1988) 241-320.

[12] Constantinescu, F., Thalhaimer, W.: Ultradistributions and quantum fields: Fourier-Laplace transforms and boundary values of analytic functions. Rept. on Math.Phys. vol.16, No 2 (1979) 167-180

[13] Fröhlich, J.: On the triviality of $\lambda\varphi_d^4$ theories and the approach to the critical point in $d \geq 4$ dimensions. Nucl. Phys. **B200** **[FS4]** (1982) 281-296.

[14] Gelfand I.M., Shilov G.E.: Prostranstva osnovnikh i obobshchennikh funktsij (Obobshchennie funktsii. Vipusk 2). Moskwa, Fizmatgiz, 1958.

[15] Gelfand, I.M., Vilenkin, N. Ya.: Nekotorie primeneniya garmonicheskogo analiza. Osnashchennie gilbertovi prostranstva. (Obobshchennie funktsii. Vipusk 4). Moskwa, Fizmatgiz, 1961.

[16] Goodman, R. W.: One-sided invariant subspaces and domains of uniqueness for hyperbolic equations. Proc. Amer. Math. Soc. **15** (1964) 653-660.

[17] Heifets, E.P.: The ϕ_4^4 classical wave equation and the construction of the quantum field as a bilinear form in the Fock space. Novosibirsk, Institute of Mathematics, 1974.

[18] Jost, R.: The general theory of quantized fields. Providence, Rhode Island, American Math. Soc., 1965.

[19] Morawetz, C. S. and Strauss, W. A.: Decay and scattering of solutions of a nonlinear relativistic wave equation, Comm. Pure Appl. Math. **25** (1972) 1-31.

[20] Morawetz, C. S. and Strauss, W.A.: On a nonlinear scattering operator, Comm. Pure Appl. Math. **26** (1972) 47-54.

[21] Osipov, E.P.: Complex structure of the manifold of solutions of classical wave equation and quantization. 6-th International Symposium on Information Theory, Part III. Moscow-Tashkent, 1984, pp. 164-166.

[22] Osipov, E.P.: Quantum interaction ϕ_4^4: the construction of quantum field defined as a bilinear form. Novosibirsk, Institute of Mathematics, TPh-205, 1994, hep-th/9602003.

[23] Osipov, E.P.: Quantum interaction ϕ_4^4: the construction of Wightman functions. Novosibirsk, Institute of Mathematics, TPh-206, 1994.

[24] Osipov, E.P.: Quantum interaction ϕ_4^4: the construction of the solution of quantum wave equation, the construction of Wightman functions. Symposium on quantization and nonlinear wave equations, MIT, Cambridge, June 1994.

[25] Osipov, E.P.: Complex structure and solutions of classical non-linear equation with the interaction u_4^4. Novosibirsk, Institute of Mathematics, TPh-207, 1995, funct-an/9602002, submitted to J. Funct. Anal.

[26] Osipov E.P.: Quantum interaction $:\phi_4^4:$. The construction of the quantum field as the solution of the nonlinear quantum wave equation. Novosibirsk, Institute of Mathematics, TPh-210, 1995.

[27] Osipov, E.P.: Complex structure and solutions of the non-linear Klein-Gordon equation with the interaction u_4^4. Novosibirsk, Institute of Mathematics, TPh-212, 1996. Submitted to Electronic Research Announcements of the AMS.

[28] Osipov, E.P.: The $:\phi_4^4:$ quantum field theory, II. Integrability of Wick kernels. In preparation.

[29] Osipov, E.P.: In preparation.

[30] Paneitz, S.: Hermitian structures on solution varieties of nonlinear relativistic wave equations, *in:* "Differential geometric methods in mathematical physics (Clausthal, 1980)", pp. 108-118, Lecture Notes in Math. 905, Springer, Berlin-New York, 1982.

[31] Paneitz, S.: Unitarization of symplectics and differential equations in Hilbert space. J. Funct. Anal. **41** (1981) 315-326.

[32] Paneitz, S.: Essential unitarization of symplectics and applications to field quantization, J. Funct. Anal. **48** (1982) 310-359.

[33] Paneitz, S.M. and Segal, I.E.: Quantization of wave equations and hermitian structures in partial differential varieties, Proc. Nat. Acad. Sci. USA **N12** (1980) 6943-6947.

[34] Paneitz, S.M., Pedersen, J., Segal, I.E., Zhou, Z.: Singular operators on boson fields as forms on spaces of entire functions on Hilbert space. J. Funct. Anal. **100** (1991) 36-58.

[35] Pedersen, J., Segal, E., Zhou, Z.: Massless ϕ_d^q quantum field theories and the nontriviality of ϕ_4^4. Nucl. Phys. **B376** (1992) 129-142.

[36] Rączka, R.: The construction of solution of nonlinear relativistic wave equation in $\lambda:\phi^4:$ theory. J. Math. Phys. **16** (1975) 173-176.

[37] Rączka, R., Strauss, W.: Analyticity properties of the scattering operator in nonlinear relativistic classical and prequantized field theories. Rept. Math. Phys. **16** No 3 (1979) 317-327.

[38] Reed, M., Simon, B.: Methods of modern mathematical physics. Vol. I. Functional analysis. New York, London, Academic Press, 1972.

[39] Reed M., Simon, B.: Methods of Modern Mathematical Physics. Vol. II. Fourier Analysis, Self-Adjointness. New York, Academic Press, 1975.

[40] Reed, M., Simon, B.: Methods of modern mathematical physics Vol. III. Scattering theory. New York, San Francisco, London, Academic Press, 1979.

[41] Segal, I.E.: Dispersion for non-linear relativistic equations. *In:* “Proc. Conf. Math. Th. El. Particles”, pp.79-108, M.I.T. Press, Cambridge, MA, 1966.

[42] Segal, I.E.: Dispersion for non-linear relativistic equations, II. Ann. Sci. École Norm. Sup. **1** (4) (1968) 459-497.

[43] Streater, R.F., Wightman, A.S.: PCT, spin and statistics, and all that. New York-Amsterdam, Benjamin 1964.

[44] Vladimirov, V.S.: Methods of the theory of functions of many complex variables. Cambridge, MIT Press 1966.